Generalized Differential Operators and their Applications

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The notion of derivation has been existing in literature since the advent of twentieth century. Many well-known algebraists like Beidar, Bell, Bergan, Bresar, Herstein, Hvala, Lanski, Ligh, Luh, Kaya, Kharchenko, Martindale, Mason, Posner, Vukman etc., have done a lot of remarkable work in this area of study which has tremendous applications in diverse parts of Mathematics.

Let R be an associative ring with centre $Z(R)$. For any pair of elements x,y in R we shall write $[x, y] = xy - yx$.

Motivated by two basic properties of a differential operator, say D (namely,

(i) $D(f_1 + f_2) = D(f_1) + D(f_2)$
(ii) $D(f_1 f_2) = D(f_1)f_2 + f_1 D(f_2)$

for any two functions f_1, f_2), the notion of derivation was introduced in rings.

Definition 1.

A mapping $d : R \to R$ *is said to be a derivation on* R *if for all* $x_1, x_2 \in R$ *the following hold:*
(i) $d(x_1 + x_2) = d(x_1) + d(x_2)$ *and*
(ii) $d(x_1 x_2) = d(x_1)x_2 + x_1 d(x_2)$.

Example 1.

The most natural example of a nontrivial derivation is the usual differentiation on the ring $F[x]$ of polynomials defined over a field F.

Example 2.

Let R be a ring and a be a fixed element of R. Define a mapping $\delta : R \longrightarrow R$ by $\delta(x) = [x, a] = xa - ax$, for all $x \in R$. Then δ is a derivation which is called the inner derivation and is denoted by I_a.

Example 3.

Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Z \right\}$. *Define a mapping* $d : R \longrightarrow R$ *such that* $d \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$

Then, it can be verified that d is a derivation on R .

Another generalization of differential operator is the Jordan derivation.

Definition 2.

An additive mapping $d : R \to R$ *is said to be a Jordan derivation if* $d(x^2) = d(x)x + xd(x)$, *holds for all* $x \in R$.

Remark 1. Every derivation is a Jordan derivation but not conversely.

Example 4.

Let R be any ring and let $a \in R$ such that $xax = 0$, but $xay \neq 0$, for some $x, y \in R, x \neq y$. Define a map $d : R \longrightarrow R$ such that $d(x) = ax$; for all $x \in R$. It can be checked that d is a Jordan derivation but not a derivation.

M. Bresar in his paper published in Glasglow Math. J. 33 (1991), 89-93 defined generalized derivation in rings.

Definition 3.

An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$, holds for all $x, y \in R$.

Example 5.

Let S be any ring and let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}$. *Define* $F : R \longrightarrow R$ *such that* $F(x) = 2e_{11}x - xe_{11}$. *Thus it can be easily seen that F is a generalized derivation with associated derivation* $d : R \longrightarrow R$ *defined by* $d(x) = e_{11}x - xe_{11}$.

Remark 2. The following example is sufficient to demonstrate that a generalized derivation need not be a derivation.

Example 6.

Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Z \right\}$. *Define* $F : R \longrightarrow R$ *by* $F\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ *and a derivation* $d : R \longrightarrow R$ *by* $d\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. *Then it can be verified that* F *is a generalized derivation but not a derivation.*

It is to remark that theory of derivations plays an important role not only in ring theory, but also in functional analysis. An extensive deep theory has been developed specially for derivations in C^*-algebras and Banach algebras, for references one may look into the following:

1. Ara, P. and Mathieu, M., *An application of local multipliers to centralizing mappings of C^* -algebras*, Quart. J. Math. Oxford 44(1993), 129-138.

2. Sinclair, A. M., *Continuous derivations on Banach algebras,* Proc. Amer. Math. Soc. 20 (1969), 166-170.

3. Vukman, J., *On derivations in prime rings and Banach algebras,* Proc. Amer. Math. Soc. 116 (1992), 877-4.

4. Vukman, J., *A result concerning derivations in Banach algebras,* Proc. Amer. Math. Soc.116 (1992), 971-975.

5. Vukman, J., *A result concerning derivations in non-commutative Banach algebras,* Glasnik Matematicki 26 (1991), 83-88.

6. Wendel, J. G., *Left centralizers and isomorphism of group algebras,* Pacific J. Math. 2 (1952), 251-261.

The study of derivations initiated long back, got impetus soon after E. C. Posner proved a very striking result in his Paper Derivations in prime rings,
Proc. Amer. Math. Soc. 8 (1957), 1093-1100.

The result which I have refered states as follows:

Theorem 1.

Let R be a prime ring admitting a nonzero derivation d such that $[d(x), x] \in Z(R)$, for all $x \in R$. Then R must be commutative.

Any derivation d on a ring R with the above property is called a **centralizing derivation** on R and in special case when $[\mathbf{d}(\mathbf{x}), \mathbf{x}] = \mathbf{0}$, for all $x \in R$, d is called a **commuting derivation** on R.

Remark 3. It is evident by the the following example that Posner's Theorem can not be proved for arbitrary rings.

Example 7.

Consider a ring $R = R_1 \times R_2$, where R_1 and R_2 are nonzero rings. R_1 is a commutative ring having a nonzero derivation d_1 and R_2 is a non-commutative ring. R is a non-commutative ring and $d : R \longrightarrow R$ defined by $d(x_1, x_2) = (d_1(x_1), 0)$ is a nonzero commuting derivation on R.

Over the last fifty years a number of Researchers extended Posner's Theorem in several ways. For references one can see the following , where further references can be found.

1. Bell, H. E. and Martindale, W. S. *Centralizing mappings of semiprime rings*, Canad. Math. Bull. 30 (1987), 92-101.

2. Brešar, M., *Centralizing mappings in Von-Neumann algebras,* Proc. Amer. Math. Soc. 111 (1991), 501-510.

3. Brešar, M., *Centralizing mappings and derivations in prime rings,* J. Algebra 156 (1993), 385-394.

4. Lanski, C., *Differential identities, Lie ideals and Posner's theorem,* Pacific J. Math. 134 (1988), 275-279.

5. Mayne J. H., *Centralizing automorphisms of prime rings*, Canad. Math. Bull. 19 (1976), 113-115.

6. Mayne J. H., *Ideals and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. 86 (1982), 211-212.

7. Miers, C.R., *Centralizing mappings of operator Algebras,* 59 (1979), 56-64.

Recently, Q. Deng defined *n*-cetralizing (*n*-commuting) mappings, a concept more general than centralizing mappings, in his paper,

On n- centralizing mappings of prime rings, Proc. R. Ir. Acad. 93 (1993), 171-176.

Definition 4.

Let n be a fixed positive integer. A mapping $f : R \longrightarrow R$ *is said to be n-centralizing (resp. n-commuting) if* $[f(x), x^n] \in Z(R)$ *(resp.* $[f(x), x^n] = 0$*), holds for all* $x \in R$*.*

Example 8.

Let $R = R_1 \oplus R_2$, where R_1 and R_2 are nonzero rings, R_2 is commutative ring. Define a map $d : R \longrightarrow R$ such that $d(x, y) = (0, y)$. Then it can be verified that d is an n-commuting derivation on R.

Example 9.

Let $R = M_2(GF(2))$, $GF(2)$ denotes the Galois field of two elements and $F : R \longrightarrow R$ be defined by $F\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha+\gamma & 0 \\ 0 & \beta+\delta \end{pmatrix}$ for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R$. Then F is a $GF(2)$- linear map. A direct computation yields that $[F(x), x^6] = 0$ for all $x \in R$ and hence F is a 6-commuting linear map.

In the mentioned paper the author generalized Posner's Theorem as follows:

Theorem 2.

Let R be a prime ring either of characteristic zero or of characteristic $> n$ and U be a nonzero left ideal of R. If R admits a nonzero derivation d such that d is n-centralizing on U, then R is commutative.

Recently Asma, Filippis and Faiza in a paper appeared in Tamsui oxford J. Information and Mathematical Sciences 28(2012), 425-436.

Proved the following result in case of a generalized derivation.

Theorem 3.

Let R be a prime ring, F a nonzero generalized derivation of R, L a nonzero Lie ideal of R, $n \geq 1$ a fixed integer such that F is n-centralizing on L. Then either $F(x) = \lambda x$ for all $x \in R$ and for some $\lambda \in C$, the extended centroid of R or R satisfies s_4, the standard identity of degree 4.

In 1998, Lanski in his paper Differential identities Lie ideals and Posner's Theorem, Pacific J. Math. 134 (1988), 275-279, obtained the following result:

Theorem 4.

If d is a derivation of a non-commutative prime ring R such that for some positive integer n, $[d(x), x]_n = [...[d(x), x], ..., x] = 0$, for all $x \in R$, then $d = 0$.

Very recently Asma, Dhara and Deepankar Das in Aequationes Math. 44 (2014), 225-238 generalized the above results as result follows:

Theorem 5.

Let R be a prime ring with its Utumi ring of quotients U, F a nonzero generalized derivation of R and L a noncentral Lie ideal of R. Suppose that $[F(u^{n_1}), u^{n_2}, u^{n_3}, ..., u^{n_k}] = 0$ *for all* $u \in L$*, where* $n_1, n_2, ..., n_k \geq 1$ *are fixed integers. Then one of the following holds:*
(i) there exists $\lambda \in C$*, the extended centroid of R such that* $F(x) = \lambda x$ *for all* $x \in R$*.*
(ii) R satisfies S_4*, the standard identity in four variables.*

Further Asma, Filippis and Faiza extended the result in case of multilinear polynomials published in
Communications in Algebra, 42 No. 9 (2014), 3699-3707.

These results are wide generalizations of Posner's Theorem and many results obtained by various authors.

In 1955, Senger and Wermer in their paper
Derivations on commutative normed algebras, Math. Ann. 129 (1955), 260-264,
established the following result:

Theorem 6.

Every continuous derivation on a commutative Banach algebra A maps A into $rad(A)$, the Jacobson radical of A.

Further, Sinclair, A. S., obtained non-commutative extension of the above thorem in his paper,
Continuous derivations on Banach algebras, Proc. Amer. Math. Soc. 20(1969), 166-170.

The results of this kind make it interesting for Researchers to study centralizing derivations in functional analysis.

The first result in this direction was obtained by Bresar and Vukman in their paper

On left derivations and related mappings, Proc. Amer. Math. Soc. 110 (1990), 7-16.

Theorem 7.

Let d be a continuous derivation of a Banach algebra A. If $[d(x), x] \in rad(A)$, for all $x \in A$, then d maps A into $rad(A)$, the Jacobson radical of A.

In 1989, Bresar and Vukman in their paper

On some additive mappings in rings with involution, Aequations mathematica 38 (1989), 178-185,
established Posner's theorem in case of Jordan derivations

defined on a ring with involution (i.e. A ring R equipped with an additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x \in R$, commonly known as $*$-ring). More precisely they proved that an additive mapping D of a $*$-ring into itself satisfying $D(x^2) = D(x)x^* + xD(x)$, for all $x \in R$ is of the form $D(x) = ax^* - xa$, for some a in R.

One can observe that there is much to be explored for such mappings in case of $*-$rings.

Derivations and their generalizations are not much studied in near rings.

Definition 5.

A left near ring N is a triple (N,+,) with two binary operations addition + and multiplication * such that*

(i) *(N,+) is a group (not necessarily abelian)*

(ii) *(N,*) is a semigroup*

(iii) $a * (b + c) = a * b + a * c$, *for all* $a, b, c \in N$.

Analogously, one can define a right near ring.

As in both the cases, the theory of near rings runs completely parallel we may consider left near rings throughout and for simplicity call them as near rings.

Example 10.

The set of all identity preserving mappings of an additive group (not necessarily abelian) into itself with pointwise addition and composition of mappings as multiplication is most natural example of a right near ring.

Example 11.

$R = \{0, a\}$ with addition and multiplication defined as follows:

$+$	0	a
0	0	a
a	a	0

$*$	0	a
0	0	a
a	0	a

It is easily checked that R is a left near ring.

For more examples one may consult: Clay, J. R. The near rings on groups of lower order, Math. Z. 104(1968), 364-371.

Definition 6.

An additive mapping $d : N \to N$ is said to be a derivation on a near ring N if
$d(xy) = xd(y) + d(x)y$, holds for all $x, y \in N$.

Example 12.

Consider $N = N_1 \oplus N_2$, where N_1 is a non-commutative near ring and N_2 is a commutative ring admitting a nonzero derivation δ. Define $d : N \longrightarrow N$ by $d((u_1, u_2)) = (0, \delta(u_2))$. Then d is a derivation on N.

Recently, it was shown by Wang in his paper Derivations in prime near rings, Proc. Amer. Math. Soc. 121 (1994), 361-366 condition is equivalent to $\mathbf{d}(\mathbf{xy}) = \mathbf{d}(\mathbf{x})\mathbf{y} + \mathbf{xd}(\mathbf{y})$, for all $x, y \in N$, which facilitates the study of derivations in near rings.

Definition 7. (Generalized derivation)

Let $d : N \longrightarrow N$ be a derivation on a near ring N. An additive mapping $F : N \longrightarrow N$ is said to be a right generalized (resp. left generalized) derivation with associated derivation d if $F(xy) = F(x)y + xd(y)$ (resp. $F(xy) = d(x)y + xF(y)$) for all $x, y \in N$ and F is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation d.

All derivations are generalized derivations.

Example 13.

Let S be a left near-ring and let

$$N = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in S \right\}.$$

Then N is a left near-ring under matrix addition and matrix multiplication. Define maps $d, F : N \longrightarrow N$ by

$$d\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ and } F\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

It can be verified that F is a right generalized derivation on N with associated derivation d.

Example 14.

Let S be a left near-ring and

$$N = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in S \right\}.$$

Then N is a left near-ring under matrix addition and matrix multiplication. Define maps $d, F : N \longrightarrow N$ by

$$d \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ and } F \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

It can be verified that F is a left generalized derivation on N with associated derivation d.

Example 15.

Let S be a left near-ring and let

$$N = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in S \right\}.$$

Then N is a left near-ring under matrix addition and matrix multiplication. Define maps $d, F : N \longrightarrow N$ by

$$d\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \text{ and } F\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

It can be checked that F is a **right generalized derivation** as well as a **left generalized derivation** on N with associated derivation d. Hence F is a **generalized derivation** on N with associated derivation d.

Theorem 8. (Posner's Second Theorem)

Let R be prime ring with characteristic 2. If d_1 and d_2 are derivations on N such that $d_1 d_2$ acts as a derivation, then either $d_1 = 0$ or $d_2 = 0$.

In 2001 Howard E. Bell and Nurcan Argac proved Posner's second theorem in case of near rings published in Algebra Colloquim 8(2001), 399-407.

Theorem 9.

Let N be a prime near ring with $2N \neq \{0\}$, and U be a nonzero semigroup ideal of N. If d_1 and d_2 are derivations on N such that $d_1 d_2$ acts as a derivation on N, then $d_1 = 0$ or $d_2 = 0$.

More recently Asma, Bell and Phool Miyan extended the aforementioned results for generalized derivations in the setting of a semigroup ideal of a prime near ring published in Afrika Mat., vol. 26, 3 (2015), 275-282.

Theorem 10.

Let N be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of N, and let F_1 and F_2 be generalized derivations on N with associated derivations d_1 and d_2. If d_1 and d_2 are not both zero and F_1F_2 acts on U as a generalized derivation with associated derivation d_1d_2, then $F_1 = 0$ or $F_2 = 0$.

Theorem 11.

Let N be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of N, and let F_1 and F_2 be generalized derivations with associated derivations d_1 and d_2. If $F_1F_2(U) = \{0\}$, then $F_1 = 0$ or $F_2 = 0$.

Theorem 12.

Let N be a 2-torsion free prime near-ring with nonzero semigroup ideal U; and let F_1 and F_2 be generalized derivations with associated derivations d_1 and d_2, such that $F_1(U^2) \subseteq U$ and $F_2(U^2) \subseteq U$. If $F_1(x)F_2(y) + F_2(y)F_1(x) = 0$ for all $x, y \in U$, then $F_1 = 0$ or $F_2 = 0$.

Theorem 13.

Let N be a 2-torsion free prime near-ring and U a nonzero semigroup ideal; and let F_1 and F_2 be generalized derivations with associated derivations d_1 and d_2, such that $F_1(x)F_2(y) + F_2(y)F_1(x) = 0$ for all $x, y \in U$.Then $F_1 = 0$ or $F_2 = 0$ if one of the following is satisfied : (a) $d_1(Z) \neq \{0\}$ and $d_2(Z) \neq \{0\}$; (b) $U \cap Z \neq \{0\}$.

Theorem 14.

Let N be a 2-torsion free prime near ring and U be a nonzero semigroup ideal of N which is closed under addition.
(i) If N has nonzero generalized derivations F_1, F_2 such that $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$, for all $x, y \in U$ and at least one of $F_1(U) \cap Z$ and $F_2(U) \cap Z$ is nonzero, then N is a commutative ring.
(ii) If N admits a nonzero generalized derivation F with associated derivation d such that $U \cap Z \neq \{0\}$ and $xF(y) + F(y)x \in Z$, for all $x, y \in U$, then N is a commutative ring.

Theorem 15.

Let N be a 2-torsion free prime near ring and U be a nonzero semigroup ideal of N which is closed under addition. Suppose N admits nonzero generalized derivations F_1 and F_2 with associated derivations d_1 and d_2 respectively such that $F_1(x)F_2(y) + F_2(y)F_1(x) \in Z$, for all $x, y \in U$ and $F_1(U) \subseteq U$ and $F_2(U) \subseteq U$. If $F_1(N) \cap Z \neq \{0\}$ or $F_2(N) \cap Z \neq \{0\}$, then N is a commutative ring.

Thank You