Laplace Transform: Properties and Applications

Professor Subuhi Khan

Department of Mathematics
Aligarh Muslim University
INDIA

Motivation

- The theory of Laplace transform also referred to as operational calculus serves as an essential tool of the mathematical background required for engineers, physicists, mathematicians and other scientists.

- The Laplace transform methods provide easy and effective means for the solutions of many problems arising in various fields of science and engineering.

Cont...

- The Laplace transform converts a function in some domain into a function in another domain, without changing the value of the function.
- The Laplace transform is used to convert complex differential equations to relatively simple equations involving polynomials. Since equations having polynomials are easier to solve. The Laplace transform makes calculations easier.

Laplace Transform

Suppose $F(t)$ is a real or complex valued function of the variable $t > 0$ and s is a real or complex parameter. Then the Laplace transform of $F(t)$ denoted by $\mathcal{L}(F(t))$ is defined by

$$\mathcal{L}(F(t)) = f(s) = \int_0^\infty e^{-st} F(t)\, dt. \qquad (1)$$

The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some value of s, otherwise it does not exist. The integral (1) converges whenever the limit of integration exists as a finite number.

Laplace Transforms of Some Elementary Functions

S.No.	Function $F(t)$	Laplace Transform $\mathcal{L}\{F(t)\} = f(s)$
1	1	$\frac{1}{s}$ (s> 0)
2	t	$\frac{1}{s^2}$ (s> 0)
3	t^n, $n = 0, 1, 2, \ldots\ldots$	$\frac{n!}{s^{n+1}}$ (s> 0)
4	e^{at}	$\frac{1}{s-a}$ (s> a)
5	$Sin\ at$	$\frac{a}{s^2+a^2}$ (s> 0)
6	$Cos\ at$	$\frac{s}{s^2+a^2}$ (s> 0)
7	$Sinh\ at$	$\frac{a}{s^2-a^2}$ (s> |a|)

1. Linearity Property

If c_1 and c_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\,\mathcal{L}\{F_2(t)\} = c_l f_1(s) + c_2 f_2(s).$$

Example

$$\mathcal{L}\{4t^2 - 3\cos 2t - 5\,e^{-t}\} = 4\,\mathcal{L}\{t^2\} - 3\,\mathcal{L}\{\cos 2t\} + 5\,\mathcal{L}\{e^{-t}\}$$

$$= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right)$$

$$= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}$$

2. Translation or Shifting Property

If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a). \tag{2}$$

Example

Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Change of Scale Property

If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Example

Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we have

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3} \frac{1}{\left(\left(\frac{s}{3}\right)^2 + 1\right)} = \frac{3}{s^2+9}$$

Applications of Laplace Transform in Engineering

The Laplace transform has applications in almost all engineering disciplines.

Analysis of Electrical Circuits

In electrical circuits, a Laplace transform is used for the analysis of linear time-invariant systems.

Analysis of Electronic Circuits

Laplace transform is widely used by electronics engineers to quickly solve differential equations occurring in the analysis of electronic circuits.

A Simple Real Life Application of Laplace Transform

A simple Laplace transform is conducted while sending signals over any two-way communication medium (FM/AM stereo, 2-way radio sets, cellular phones).

When information is sent over medium such as cellular phones, they are first converted into time-varying wave, and then it is super-imposed on the medium. In this way, the information propagates.

At the receiving end, to decipher the information being sent, medium wave's time functions are converted to frequency functions.

THANK YOU